Bounds for Holonomic Sequences and Guaranteed Precision Evaluation

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Bounds

Motivation

- Baxter permutations
  - $(n+2)(n+3)B_n = (7n^2+7n-2)B_{n-1} + 8(n-1)(n-2)B_{n-2}$, $B_0 = B_1 = 1$
  - $B_n \leqslant (n+8)^8 8^n$
- $t_k = \dfrac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k}}$
  - $\dfrac{12}{640320^{3/2}} \sum_{k=0}^{\infty} t_k = \dfrac{1}{\pi}$ (Chudnovsky^2 1989)
  - $\left| \dfrac{640320^{3/2}}{12\pi} - \sum_{k=0}^{n-1} t_k \right| \leqslant (0{,}1n^4 + 0{,}5n^3 + 1{,}5n^2 + 2{,}1n + 1)\alpha^n$

    where $\alpha = \dfrac{1}{151931373056000} \simeq 0,66 \cdot 10^{-14}$

Asymptotic Behaviour of Recurrences

$$y_{n+s} + a_{s-1}(n)\, y_{n+s-1} + \cdots + a_1(n)\, y_{n+1} + a_0(n)\, y_n = 0$$

- well-defined
- reversible
- $a_k(n) \to a_k^\infty$ as $n \to \infty$

$$y_{n+s} + a_{s-1}^\infty\, y_{n+s-1} + \cdots + a_1^\infty\, y_{n+1} + a_0^\infty\, y_n = 0$$

- characteristic polynomial $X^s + a_{s-1}^\infty X^{s-1} + \cdots + a_0^\infty$

Theorem (Poincaré-Perron)

Let $\alpha_1, \ldots, \alpha_s$ be the roots of the characteristic polynomial.

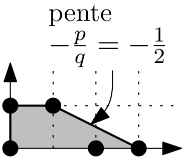
1. *There is a basis of solutions $(y^{[k]})$ of the recurrence above s.t.* $\limsup_{n\to\infty} |y_n^{[k]}|^{1/n} = \alpha_k$.
2. *If the roots of the characteristic polynomial have distinct absolute values, then we even have* $y_{n+1}^{[k]}/y_n^{[k]} \to \alpha$.

Asymptotic Behaviour of Recurrences II

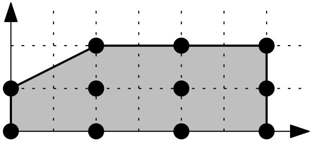
Newton Polygon

$$p_s(n)\, y_{n+s} + p_{s-1}(n)\, y_{n+s-1} + \cdots + p_1(n)\, y_{n+1} + p_0(n)\, y_n = 0$$

- $|p_k(n)/p_s(n)| \to \infty$ for some k (or $p_k(n)/p_s(n) \to 0$ for all k)



$$u_n = \psi_n y_n$$
$$(n+q)^p \psi_{n+q} = \psi_n$$



$y_{n+3}+y_{n+2}+ny_{n+1}+(n+1)y_n=0$

$(n+6)(n+4)u_{n+6}+2(n+4)(n+1)u_{n+4}-(n^2-n-5)u_{n+2}-(n+1)u_n=0$

Conclusion : y_n "$\simeq$" $n!^{p/q}\alpha^n$

Objectif

'Tight' Bounds

Input Recurrence + Initial values

$$\{p_s(n)\, y_{n+s} + \cdots + p_0(n)\, y_n = 0, \quad y_0 = \ldots\}$$

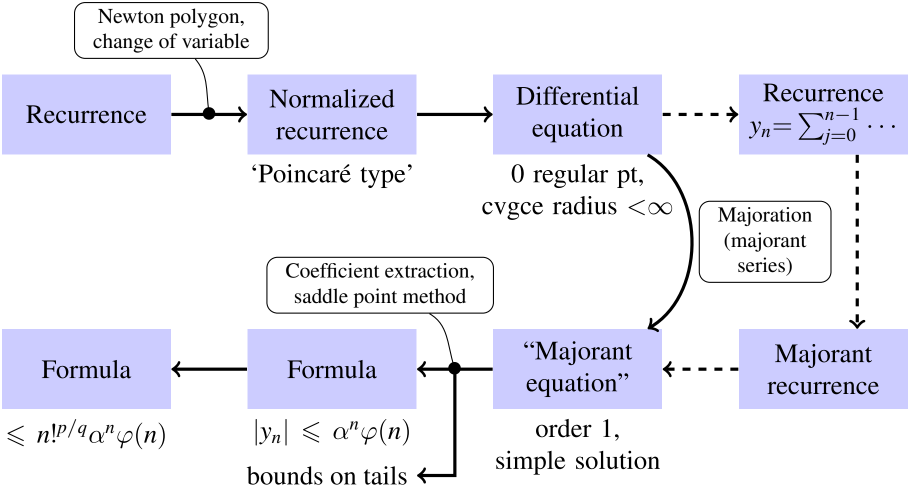
Output $|y_n| \leqslant n!^{p/q}\, \alpha^n\, \varphi(n)$

with φ subexponential, i.e. $\varphi(n) = e^{o(n)}$

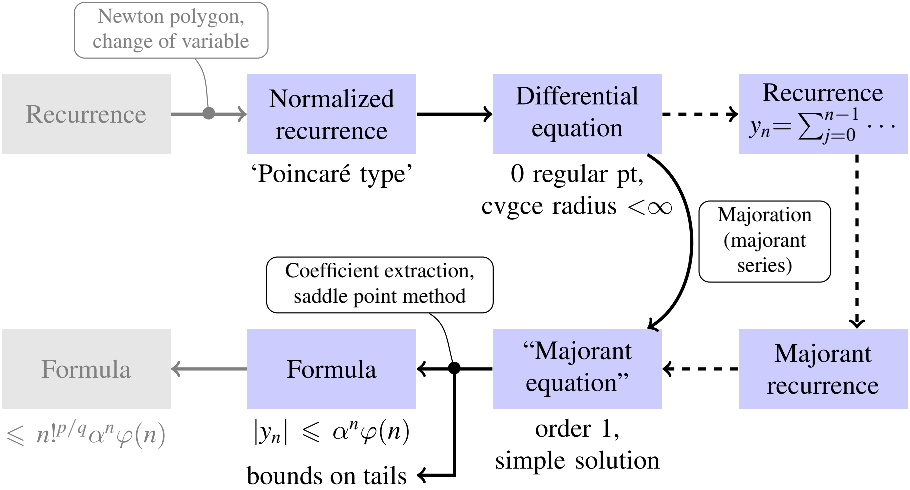
- correct bound
- for generic initial values: optimal p/q and α, $\varphi(n) = n^{O(1)}$ when possible

We use the asymptotic behaviour of the solutions as a guide.

Strategy


Newton polygon, change of variable
Recurrence
Normalized recurrence
'Poincaré type'
Differential equation
0 regular pt, cvgce radius $<\infty$
Recurrence $y_n = \sum_{j=0}^{n-1} \cdots$
Majoration (majorant series)
Coefficient extraction, saddle point method
Formula
$\leqslant n!^{p/q}\alpha^n\varphi(n)$
Formula
$|y_n| \leqslant \alpha^n\varphi(n)$
bounds on tails
"Majorant equation"
order 1, simple solution
Majorant recurrence

Strategy



Newton polygon, change of variable
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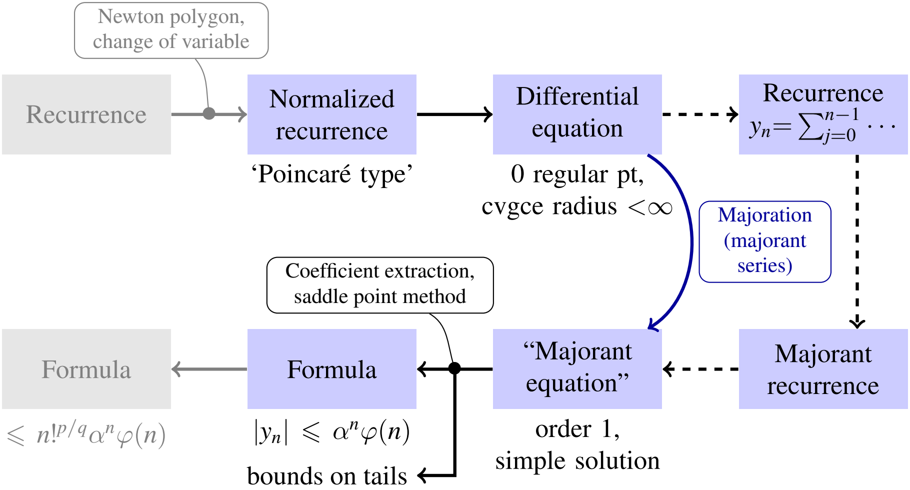
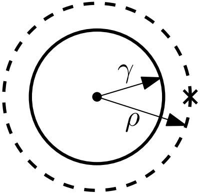


Preserve: convergence radius, nature of the dominant singularity

Majorant Series

Cauchy-Kovalevskaya Method: Warm-Up

$y'(z) = a(z)\, y(z)$ $\quad a(z)$ analytic for $|z| < \rho$



- $(n+1)y_{n+1} = \sum_{j=0}^{n} a_j y_{n-j}$
- Let M be s.t. $\forall j, |a_j| \leqslant M/\gamma^j$, consider $(n+1)g_{n+1} = \sum_{j=0}^{n} M\gamma^{-j} g_{n-j}$
- We then have $g'(z) = M(1 - z/\gamma)^{-1} g(z)$ thus $g(z) \propto (1 - z/\gamma)^{-\gamma M}$
- By induction $|y_0| \leqslant g_0 \implies \forall n, |y_n| \leqslant g_n$
- But $g(z)$ is analytic for $|z| \leqslant \gamma$ $(\to \rho)$. $\square$

J. van der Hoeven. Fast evaluation of holonomic functions near and in regular singularities. 2001.

J. van der Hoeven. Majorants for formal power series. 2003.

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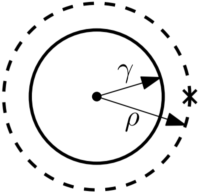

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Difficulties

- We want $\gamma = \rho$
- 0 reg. singular

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Majorant Differential Equation

$$z^r y^{(r)} - c_{r-1} z^{r-1} y^{(r-1)} - \cdots - c_0 y = z^r a^{[r-1]}(z) y^{(r-1)} + z^{r-1} a^{[r-2]}(z) y^{(r-2)} + \cdots + z a^{[0]}(z) y$$

$$q(n)\, y_n = \sum_{j=0}^{n-1} \sum_{k=0}^{r-1} a^{[k]}_{n-1-j}\, j^{\underline{k}}\, y_j$$

$$|a_n^{[k]}| \leqslant [z^n] \frac{M_k}{(1-\alpha z)^{r-k+T}}$$

$(T = \text{irregularity})$

$$g_n = \frac{K}{n} \sum_{j=0}^{n-1} \binom{(n-1-j)+T}{T} \alpha^{n-j} g_j \qquad g'(z) = \frac{\alpha K}{(1-\alpha z)^{1+T}} g(z)$$

Majorant Differential Equation

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$$|a^{[k]}_n| \leqslant [z^n] \frac{M_k}{(1-\alpha z)^{r-k+T}}$$

(T = irregularity)

For n large enough:

$$|y_n| \leqslant \frac{1}{|q(n)|} \sum_{j=0}^{n-1} \left(\sum_{k=0}^{r-1} M_k \binom{(n-1-j)+(r-k-+T-1)}{r-k+T-1} j^{\underline{k}} \right) \alpha^{n-1-j}\, |y_j|$$

$$\leqslant \underbrace{\frac{M n^{r-1}}{|q(n)|}}_{\leqslant K/n} \sum_{j=0}^{n-1} \binom{(n-1-j)+T}{T} \alpha^{n-j}\, |y_j|$$

$q(n) = n^r + \cdots$

$$g_n = \frac{K}{n} \sum_{j=0}^{n-1} \binom{(n-1-j)+T}{T} \alpha^{n-j} g_j \qquad g'(z) = \frac{\alpha K}{(1-\alpha z)^{1+T}} g(z)$$

'Tight' Majorant Series

Majorant equation: $g'(z) = \dfrac{\alpha K}{(1-\alpha z)^{1+T}} g(z)$

$\boxed{T = 0}$ $g(z) = \dfrac{A}{(1-\alpha z)^K}$

$$|y_n| \leqslant g_n = A \binom{n+K-1}{K-1} \alpha^n$$

$\boxed{T > 0}$ $g(z) = A \exp \dfrac{K/T}{(1-\alpha z)^T}$

$$|y_n| \leqslant g_n \leqslant A \exp\left(C n^{T/(T+1)}\right) \alpha^n \qquad \text{for } n \geqslant N$$

(saddle point method)

Extra: Bounds on tails, on derivatives…

Numerical Evaluation

High-Precision Evaluation of Holonomic Functions

Connection with Bound Computations

- To compute $y(z_1)$ to a (user-given) precision $\varepsilon = 2^{-d}$
  1. Compute $N = O(d)$ s.t. $\left| y(z_1) - \sum_{n=0}^{N-1} y_n z_1^n \right| \leqslant \frac{\varepsilon}{2}$ (part I)
  2. Compute $\sum_{n=0}^{N-1} y_n z_1^n$
- High precision $d \to \infty$
- 'Tight' bound on the y_n's
  - only $o(d)$ superfluous digits (otherwise: $O(d)$)
  - majorant series independent of z_1

High Precision

Algorithms, Implementations

High precision: Complexity w.r.t. the number of digits

"The previous strategy (sum the series) is competitive"

Algorithms which are

- General (all holonomic functions)
- Quasi-linear : time complexity $d(\log d)^{O(1)}$ for d digits
- Simple & efficient (in practice)
- Actually used… in special cases only!

Now: Examples of what is available

D.V. and G.V. Chudnovsky. Computer algebra in the service of mathematical physics and number theory. 1990.

NumGfun

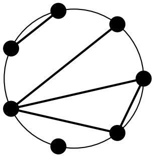
- A Maple package for the symbolic-numeric manipulation of holonomic functions
  - Guaranteed numerical evaluation
  - Bounds
  - ...
- Version 0.1 available, version 0.2 to come soon (Experimental!)
- Integration of some parts into gfun in progress

Bruno Salvy and Paul Zimmermann, Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable, 1994.

J. van der Hoeven. Fast evaluation of holonomic functions. 1999.

Binary Splitting for Sequences (I)

Motzkin Numbers



0, 1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798, 15511, 41835, 113634, 310572, ...

$$(n+3)\,M_{n+2} = 3n\,M_n + (2n+3)\,M_{n+1},$$
$$M_0 = 0, M_1 = M_2 = 1$$

$$M_{1\,000\,000} = \begin{array}{l} 87836485521410228205552857212867952 \\ 60648460114018772686310027332206011651992742068 \\ 95017531901406553089345501470120232183076893776 \\ 76219223691237769669136651142176793088580998640 \\ 24791593930900669539159753966399354360360024084 \\ 835778 \ldots\ 67840785185707760882612226992209195 25 \\ 44768602806558705745804408930594940932105099980 \\ 80763012645020992166911388664219549747372475451 \\ 13677895449716717989937706488976239581832306432 \\ 74956942565741376149791829585290393680786291940 \end{array}$$

(477 112 digits)

Binary Splitting for Sequences (II)

One Million Decimal Digits of π

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}} \qquad (\text{Chudnovsky}^2 \ 1989)$$

$\pi \simeq$ 3,14159265358979 23846264338327950 28841971693993751 05820974944592307 81640628620899862 80348253421170679 82148086513282306 64709384460955058 22317253594081284 811174502841027 01 …

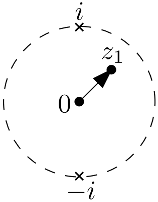
. . .

… 879 613 033 1164 6283 9963 46460 422090106105779458151

Elementary Functions, Special Functions

$$(1+z^2)\ \arctan''(z) + 2z\ \arctan'(z) = 0$$

$$\arctan \frac{3(1+i)}{5} \simeq 0{,}670782196758950644190815337$$
4705632571369265547562721682009119775363456
2788546268206648547182112134208947460355580
1433079787592299964529081793221227836458496
7241027751816658681028242709786087804231203
5059588657436137542728611075919334091735855
+ 0,4313775209217135982596553539683059915248
7122502784763704416333662458132714904677846
9188664848592351371193308077157250027646988
5281752378714171283456698686337133570545945
8746821430812351884522098343403327937148536
338890142864171080500321 i

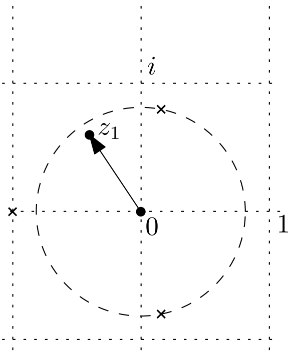


General Holonomic Functions (of one variable)

$$(z+1)(3z^2-z+2)\,y''' + (5z^3+4z^2+2z+4)y'' \\ +(z+1)(4z^2+z+2)y' + (4z^3+2z^2+5)y = 0$$

$$y(0)=0,\ y'(0)=i,\ y''(0)=0$$

$y(z_1) \simeq -\,$0,5688220713892109968232887489539
40401816728372266594043883320346219592758
12320494797058201136707120728488174753296
40179618640233165335353913821228176742066
38746845195076195216482627052648481989147
$-\,$0,41951120825888216814674495005568322636
04890369475390958159560577151580169021584
69436992399704818660023662419290957376458
10730416775833847769588392648233263560262
18036663454753771692569046113725631 i

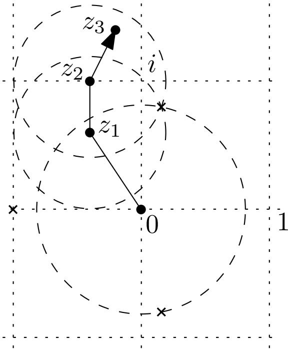


$$z_1 = \frac{-2+3i}{5}$$

Numerical Analytic Continuation

$$(z+1)(3z^2-z+2)\,y''' + (5z^3+4z^2+2z+4)y'' \\ +(z+1)(4z^2+z+2)y' + (4z^3+2z^2+5)y = 0$$

$$y(0)=0, y'(0)=i, y''(0)=0$$



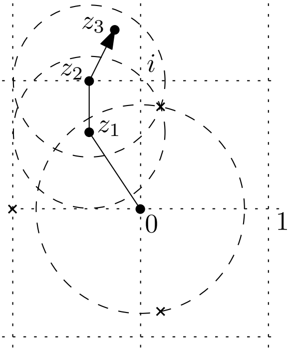
$y(z_3) \simeq -1{,}5598481440603221187326507993405$
$9338934133466448795950045370633754599013 0$
$2359572361012065551669069709899240095229 3$
$0251611714754471345284564264496647625428 8$
$7666223763565716341513188606343080316103 9$
$-\ 0.71077649435126718436732868786933143977$
$5904747961810404577707695459155140694934 5$
$1433687429553335664986950937759284160623 9$
$8437391943410973508428254938741106987743 7$
$70372320294299156084733705293726504\ i$

$$z_3 = \frac{-1+7i}{5}$$

Numerical Connection Between Ordinary Points

$$(z+1)(3z^2-z+2)\,y''' + (5z^3+4z^2+2z+4)y'' + (z+1)(4z^2+z+2)y' + (4z^3+2z^2+5)y = 0$$

$$\begin{bmatrix} y(z_3) \\ y'(z_3) \\ y''(z_3) \end{bmatrix} = \underbrace{\begin{bmatrix} \ \end{bmatrix}}_{\begin{bmatrix} 1{,}229919181 + 1{,}222484838i & -0{,}710776494 + 1{,}559848144i & -1{,}680450593 + 0{,}8612944465i \\ 2{,}192415163 - 0{,}982260350i & 1{,}428307159 + 1{,}237636972i & 1{,}683681888 + 1{,}443224767i \\ -0{,}810105380 - 0{,}813018670i & 0{,}949416034 - 0{,}368995278i & -0{,}309094585 - 0{,}032241130i \end{bmatrix}} \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix}$$



$$z_3 = \frac{-1+7i}{5}$$

Evaluation Points of Large Bit Size

Bit burst Evaluation

$$\begin{aligned}\mathrm{erf}(\pi) \simeq\ & 0.99999112385363235839473162078120294471238208 15\\ & 1287659904758639164678439426196498460278504541782613310\\ & 0604326482152030660441196387585407489394338729142916313\\ & 2555230902334047429212609807578643285046857228864728035\\ & 3074866062036004350772927038034048195719630178507694248\\ & 4951063443190106356178078634699387973616755577593078576\\ & 7867193730580658008654893571733600902958925087790354763\\ & 1634821321290934135517729080384812555377261445353232562\\ & 6651433607961144658060331385205962860463925296434774976\\ & 4667106060908609383010103929356543447438130957966770981\\ & 9560099884058213492947592606412648383713291083934904913\\ & 3976893748259243076371780227275937091363807381587573107\end{aligned}$$

(Bounds implementation not yet finished in this case)

Summary

1. Bounds with the ‘right’ exponential growth rate for holonomic sequences / Majorant series with the ‘right’ convergence radius for holonomic functions
2. Code for guaranteed high-precision numerical analytic continuation

Related Problems

- (more) direct description of the majoration technique
- tighter and more general bounds
- interaction with other bound computation techniques
- *really* efficient implementation of binary splitting
- complexity: constant factor, dependence on the function

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Merci ! / Thank you!