Evaluation of Ai(x) with Reduced Cancellation

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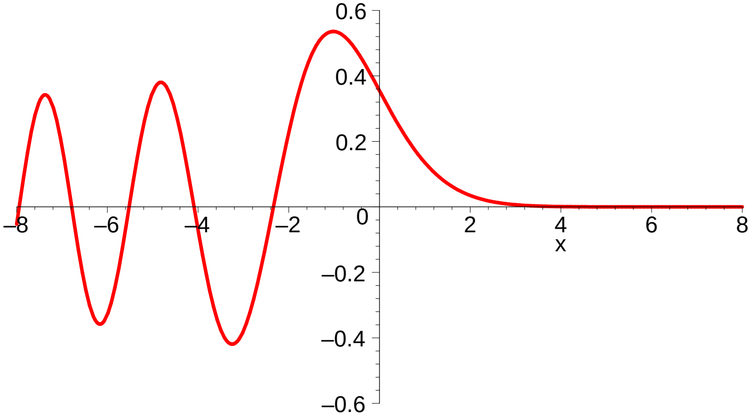
The Airy Function $\mathrm{Ai}(x)$



$$\mathbf{Ai}''(x) = x\,\mathbf{Ai}(x) \qquad \mathrm{Ai}(0) = \frac{1}{3^{2/3}\,\Gamma(2/3)} \qquad \mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\,\Gamma(1/3)}$$

Multiple-Precision Evaluation for $x > 0$

Standard Approach

"Small" x:

Taylor Series at 0

- catastrophic cancellation for moderately large x
- need $p_{\mathrm{work}} \gg p_{\mathrm{res}}$

"Large" x:

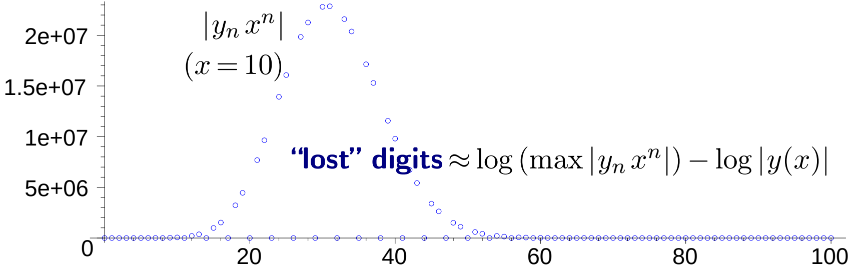
Asymptotic Expansion at ∞

This talk

New evaluation algorithm for "small" x with $p_{\mathrm{work}} \approx p_{\mathrm{res}}$

Complete error analysis

Catastrophic Cancellation

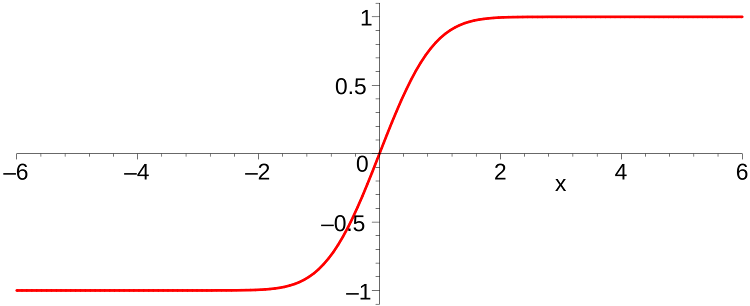


"lost" digits $\approx \log(\max |y_n x^n|) - \log |y(x)|$

$$\mathrm{Ai}(x) = A - B\,x + \frac{A}{6}\,x^3 - \frac{B}{12}\,x^4 + \frac{A}{180}\,x^6 - \frac{B}{504}\,x^7 + \frac{A}{12960}x^9 - \cdots$$

Another Example

The Error Function



$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left(x - \frac{1}{3} x^3 + \frac{1}{10} x^5 - \frac{1}{42} x^7 + \frac{1}{216} x^9 - \cdots \right)$$

catastrophic cancellation

But...

Series Expansions

7.1.5 $$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

7.1.6 $$= \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{1 \cdot 3 \ldots (2n+1)} z^{2n+1}$$

[Abramowitz & Stegun 1972, p. 297]

Algorithm

1. Compute $\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{1 \cdot 3 \cdots (2n+1)}$ positive terms, no cancellation
2. Compute $\exp(x^2)$
3. Divide

Where

does this formula

come from?

The Gawronski-Müller-Reinhard Method

Or: How Complex Analysis "explains" the previous trick

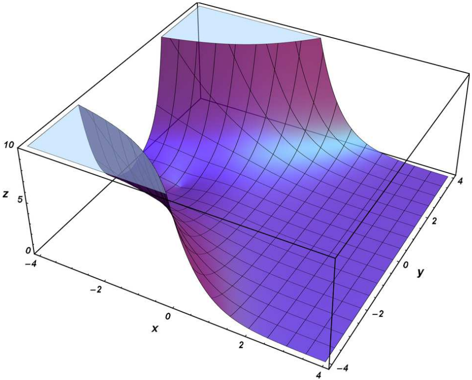
Idea: **Find F and G** such that

1. $y(x) = \dfrac{G(x)}{F(x)}$
2. F and G computable with little cancellation

W. Gawronski, J. Müller, M. Reinhard. *SIAM J. Num. An.*, 2007

M. Reinhard. Phd thesis, Universität Trier, 2008

Asymptotics



$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}\, z^{3/2}\right)}{2\,\sqrt{\pi}\, z^{1/4}}$$

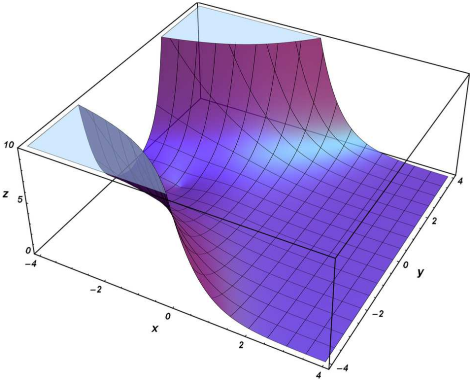
as $z \to \infty$

in any sector

$\{z \in \mathbb{C} | -\varphi < \arg z < \varphi\}$

with $\varphi > 0$

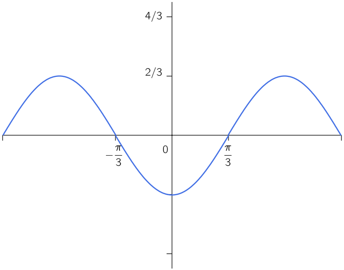
Asymptotics



$$\mathrm{Ai}(z) \sim \frac{\exp\left(-\frac{2}{3}\, z^{3/2}\right)}{2\,\sqrt{\pi}\, z^{1/4}}$$

Order $\rho = 3/2$

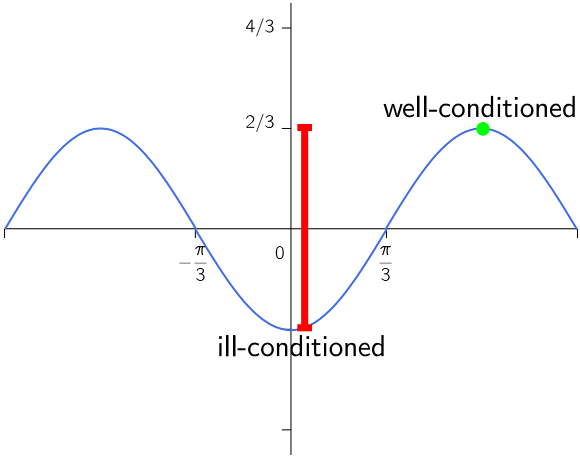
Indicator $h(\theta) = -\dfrac{2}{3}\cos\dfrac{3\,\theta}{2}$



$$|\mathrm{Ai}(r\, e^{i\theta})| \approx \exp\left(\boldsymbol{h}(\theta)\, r^{\boldsymbol{\rho}}\right)$$

for large r

Lost in Cancellation



$$\textbf{lost digits} \approx \log\left(\max_n |y_n\,(r\,e^{i\theta})^n|\right) - \log|y(r\,e^{i\theta})| \approx r^\rho\,(\boldsymbol{\max h - h(\theta)})$$

The GMR Method

$$\begin{cases} |F(z)| \approx \exp\left(\boldsymbol{h_F(\theta)}\, r^\rho\right) \\ |G(z)| \approx \exp\left(\boldsymbol{h_G(\theta)}\, r^\rho\right) \end{cases} \quad \Rightarrow \quad \left|\frac{G(z)}{F(z)}\right| \approx \exp\Big[\underbrace{(\boldsymbol{h_G(\theta)} - \boldsymbol{h_F(\theta)})}_{h_{G/F}(\theta)}\, r^\rho\Big]$$

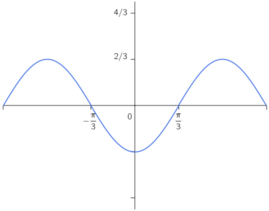
Idea (refined): look for

- an auxiliary series F,
- a modified series $G = y\,F$,

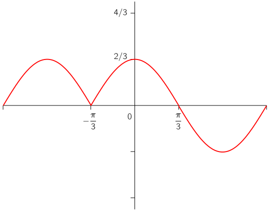
both of order ρ,

such that h_F and $h_G \approx$ their max for $\theta = 0$

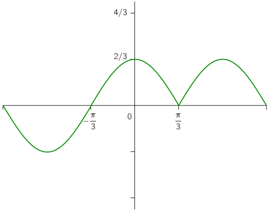
Indicators



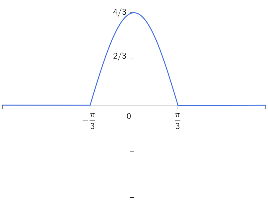
$\mathrm{Ai}(x)$



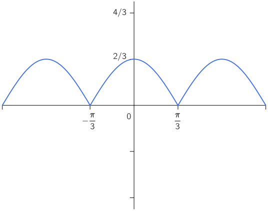
$\mathrm{Ai}(j^{-1}\,x)$



$\mathrm{Ai}(j\,x)$



$\boldsymbol{F(x)} = \mathrm{Ai}(j\,x)\,\mathrm{Ai}(j^{-1}\,x)$



$\boldsymbol{G(x)} = \mathrm{Ai}(x)\,\mathrm{Ai}(j\,x)\,\mathrm{Ai}(j^{-1}\,x)$

How do we

evaluate

the auxiliary series?

Computer Algebra to the Rescue

A function y is **D-finite** (holonomic) when it satisfies a linear ODE with polynomial coefficients.

Examples: $\mathbf{Ai(x)}, \exp(x), \mathrm{erf}(x)$... $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$

If $f(x)$, $g(x)$ are D-finite, then:

- $f(x) + g(x)$ and $f(x) \cdot g(x)$ too

  $F(x) = \mathrm{Ai}(j\,x) \cdot \mathrm{Ai}(j^{-1}\,x)$ $F'''(x) = 4\,x\,F'(x) + 2\,F(x)$

- The Taylor coefficients of $f(x)$ obey a linear recurrence relation with polynomial coefficients

  $F(x) = \sum_{n=0}^{\infty} F_n\,x^n$ $F_{n+3} = \frac{2\,(2\,n+1)}{(n+1)\,(n+2)\,(n+3)}\,F_n$

The Auxiliary Series $F(x)$

D-Finiteness

$$\boldsymbol{F_{n+3}} = \frac{2\,(2\,n+1)}{(n+1)\,(n+2)\,(n+3)}\,\boldsymbol{F_n}$$

$$F_0 = \frac{1}{3^{4/3}\,\Gamma\!\left(\frac{2}{3}\right)^2} \qquad F_1 = \frac{1}{2\,\sqrt{3}\,\pi} \qquad F_2 = \frac{1}{3^{2/3}\,\Gamma\!\left(\frac{1}{3}\right)^2}$$

- Two-term recurrence $\Rightarrow$ Easy to evaluate
- Obviously $F_n > 0$ $\Rightarrow$ Minimal cancellation

The Modified Series $G(x)$

$$G(x) = \mathrm{Ai}(x)\, F(x) = \sum_{n=0}^{\infty} G_n\, \boldsymbol{x^{3n}}$$

D-Finiteness

$$\boldsymbol{G_{n+2}} = \frac{10\,(n+1)^2\, \boldsymbol{G_{n+1}} - \boldsymbol{G_n}}{(n+1)\,(n+2)\,(3\,n+4)\,(3\,n+5)}$$

$$G_0 = \frac{1}{9\,\Gamma\left(\frac{2}{3}\right)^3} \qquad G_1 = \frac{1}{18\,\Gamma\left(\frac{2}{3}\right)^3} - \frac{1}{3\,\Gamma\left(\frac{1}{3}\right)^3}$$

$$\begin{aligned} G(x) = {} & 0.44749 \cdot 10^{-1} + 0.50371 \cdot 10^{-2}\, x^3 + .14053 \cdot 10^{-3}\, x^6 \\ & + .17388\, 10^{-5}\, x^9 + .12091 \cdot 10^{-7}\, x^{12} + .53787 \cdot 10^{-10}\, x^{15} + \cdots \end{aligned}$$

Observe that $\boldsymbol{G_n > 0}$ (proof?)

Bad News

The recursive computation of G_n is

unstable

(G_n is a minimal solution of the recurrence)

The computation of the **sum** $\sum_{n=0}^{\infty} G_n x^n$ is stable (no cancellation)

All Is Not Lost

Miller's **backward recurrence** method allows one to compute minimal solutions in a numerically stable way

Final Algorithm

1. Compute error bounds, choose working precision (how?)
2. Compute $F(x)$ by direct recurrence
3. Compute $G(x)$ using Miller's method
4. Divide

Numerically stable in practice (proof?)

I didn't actually

prove

anything

Making the Analysis Rigorous

- Prove that (G_n) is a minimal solution
  [⇒ Miller's method works]
- Prove that $G_n \geqslant 0$
  [⇒ no cancellation]

Main issue:
need **bounds on G_n**

- Bound the tails of the series F and G
- Bound the roundoff errors in $\sum F_n x^n$
- Bound the method error of Miller's algorithm
- Bound additional roundoff errors due to Miller's method [M&vdS 1976]

R.M.M. Matthiej & A. van der Sluis, *Numerische Mathematik*, 1976

Controlling G_n

Main Technical Lemma

$$G_n \sim \gamma_n = \frac{1}{4\sqrt{3}\,\pi\,9^n\,n!^2} \quad \text{with} \quad \left|\frac{G_n}{\gamma_n} - 1\right| \leqslant 2.4\,n^{-1/4} \quad \text{for all} \quad n \geqslant 1$$

Corollary: $G_n > 0$ (for large n, then for all n)

Idea of the proof

- $G_n = \dfrac{1}{2\,\pi\,i} \oint \dfrac{G(z)}{z^{3n+1}}\,\mathrm{d}z$
- saddle-point method
- $\mathrm{Ai}(z) \sim \dfrac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}\,z^{1/4}}$ + error bound

Conclusion

Summary

- New well-conditioned formula for $\mathrm{Ai}(x)$, obtained by an extension of the GMR method
- Rigorous error analysis on this example
- Ready-to-use multiple-precision algorithm for $\mathrm{Ai}(x)$

  implementation & suppl. material at `http://hal.inria.fr/hal-00767085`

Next question: How much of this is specific to $\mathrm{Ai}(x)$?

- Entire function
- Ability to find auxiliary series
- D-finiteness [constraints on the order of the recurrences?]
- Asymptotic estimate with error bound

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